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THE STABILIZATION OF CONTROLLED SYSTEMS WITH A GUARANTEED ESTIMATE OF THE CONTROL QUALITY†

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The problem of stabilizing motion in controlled systems with a guaranteed estimate of the control quality is considered. It arises from the optimal stabilization problem when the conditions on the cost functional are relaxed: no minimization of this functional is required, it is only necessary for it not to exceed a certain limit. This enables the class of solvable problems to be extended compared to the class of optimal stabilization problems. The solution of the problem is based on Lyapunov's direct method using Lyapunov functions with derivatives of constant sign. Some of the results are new even in the case of the optimal stabilization problem. The following examples are considered: a holonomic mechanical system with time-dependent Lagrangian, a controlled linear mechanical system and the problem of using the gravitational moment to stabilize the controlled plane rotational motion of a satellite in an elliptic orbit. © 1997 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

We consider a controlled system, the motion of which is described by the system of differential equations

$$\dot{\mathbf{x}} = \mathbf{X}(t, \mathbf{x}, \mathbf{u}) \tag{1.1}$$

where $\mathbf{x} = (x_1, \dots, x_n)$ is an n-vector in the real linear space R^n with norm $\|\mathbf{x}\|$, $\mathbf{u} = (u_1, \dots, u_r) \in R^r$. The right-hand side $\mathbf{X}(t, \mathbf{x}, \mathbf{u})$ ($\mathbf{X}(t, 0, 0) = 0$ in (1.1), which is defined for a class $U = \{\mathbf{u}(t, \mathbf{x}) : \mathbf{u}(t, 0) = 0\}$ of control actions $u(t, \mathbf{x}) \in C(G)$, $G = R^+ \times \Gamma$ ($R^+ = [0, +\infty[$, $\Gamma = \{\|\mathbf{x}\| < H, H = \text{const} > 0\}$), is continuous and satisfies the conditions for the existence and uniqueness of solutions in G.

Let the integral

$$I = \int_{t_0}^{\infty} W(t, \mathbf{x}[t], \mathbf{u}[t])dt \tag{1.2}$$

serve as an estimate of the control quality in this system for the transient subject to control $\mathbf{u}[t]$ along the corresponding trajectory $\mathbf{x}[t]$ of (1.1). The integrand $W(t, \mathbf{x}, \mathbf{u})$ in (1.2) is, in the general case, a non-negative continuous function defined in G for $\mathbf{u} \in U$.

Let us state the stabilization problem with guaranteed estimate of the control quality.

Definition. A control action $\mathbf{u} = \mathbf{u}^0(t, \mathbf{x})$ is said to be stabilizing with a guaranteed estimate of quality $P(t, \mathbf{x})$ if it ensures that the unperturbed motion $\mathbf{x} = 0$ of (1.1) is asymptotically stable and the inequality

$$I = \int_{t_0}^{\infty} W(t, \mathbf{x}^0[t], \mathbf{u}^0[t])dt \leqslant P(t_0, \mathbf{x}_0) \tag{1.3}$$

holds for each controlled motion $\mathbf{x}^0(t)$, $\mathbf{x}^0(t_0) = \mathbf{x}_0$.

2. ADDITIONAL ASSUMPTIONS AND CONSTRUCTIONS

Suppose that for some $\mathbf{u}^0(t, \mathbf{x}) \in U$ the right-hand side $\mathbf{X}^0(t, \mathbf{x}) = \mathbf{X}(t, \mathbf{x}^0, \mathbf{u}^0(t, \mathbf{x}))$ of (1.1) is bounded on each compact set and satisfies the Lipschitz condition uniformly in $\mathbf{x}$ with respect to t, that is, for any compact set $K \subset \Gamma$ there are two constants $\lambda_K = \lambda(K)$ and $\nu_K = \nu(K)$ such that

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$$\|\mathbf{X}^0(t,\mathbf{x})\| \leq \lambda_K, \quad \mathbf{X}\|^0(t,\mathbf{x}_2) - \mathbf{X}^0(t,\mathbf{x}_1)\| \leq \nu_K \|\mathbf{x}_2 - \mathbf{x}_1\| \tag{2.1}$$

Then $\mathbf{X}^0(t, \mathbf{x})$ satisfies the precompactness conditions in G in some functional space F_Φ [1] and with the system of equations (1.1) $\dot{\mathbf{x}} = \mathbf{X}^0(t, \mathbf{x})$ one can associate [1] a family of limit systems $\dot{\mathbf{x}} = \Phi(t, \mathbf{x})$ for which the functions $\Phi(t, \mathbf{x})$ are given by

$$\Phi(t,\mathbf{x}) = \frac{d}{dt}\left(\lim_{t_n\to\infty} \int_0^\infty \mathbf{X}^0(t_n+\tau,\mathbf{x})d\tau\right)$$

Suppose that the integrand $W^0(t, \mathbf{x}) = W(t, \mathbf{x}, u^0(t, \mathbf{x}))$ in (1.2) for $\mathbf{u}^0(t, \mathbf{x}) \in U$ satisfies similar conditions

$$\|W^0(t, \mathbf{x})\| \leq \eta_K, \quad \|W^0(t, \mathbf{x}_2) - W^0(t, \mathbf{x}_1)\| \leq \mu_K\| \mathbf{x}_2 - \mathbf{x}_1\| \tag{2.2}$$

where $\eta_K = \eta(K)$ and $\mu_K = \mu(K)$ are constants which exist for every compact set $K \subset \Gamma$. Then, by analogy, $W^0(t, \mathbf{x})$ satisfies the precompactness conditions in G in some functional space E_Ω and one can associate with it a family of limit functions $\Omega(t, \mathbf{x})$ defined by

$$\Omega(t,\mathbf{x}) = \frac{d}{dt}\left(\lim_{t_n\to\infty} \int_0^t W^0(t_n+\tau,\mathbf{x})d\tau\right)$$

Following [2] we introduce

$$B[V,t,\mathbf{x},\mathbf{u}] = \frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \mathbf{x}}\right)^T \mathbf{X}(t,\mathbf{x},\mathbf{u}) + W(t,\mathbf{x},\mathbf{u})$$

Continuous monotonically increasing functions in the section $[0, H]$ such that $\alpha(0) = 0$, that is, Hahn type functions [3], will be denoted by $\alpha(\|\mathbf{x}\|)$.

3. BASIC RESULTS

We shall present a solution of the above problem on stabilization with a guaranteed estimate of quality based on Lyapunov's direct method.

Theorem 1. Suppose that for system (1.1) with control quality estimate (1.2) a Lyapunov function $V(t, \mathbf{x}) \in C^1(G)$ and a control function $\mathbf{u} = \mathbf{u}^0(t, \mathbf{x}) \in U$ exist such that the following conditions hold

1. $V(t, \mathbf{x})$ is a positive-definite and admits of an infinitesimal upper bound, $\alpha_1(\|\mathbf{x}\|) \leq V(t, \mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|)$;
2. $W(t, \mathbf{x}, \mathbf{u}^0(t, \mathbf{x}))$ is a positive-definite function such that $W(t, \mathbf{x}, \mathbf{u}^0(t, \mathbf{x})) \geq \alpha_3(\|\mathbf{x}\|)$;
3. $B[V, t, \mathbf{x}, \mathbf{u}^0(t, \mathbf{x})] \leq 0$.

Then $\mathbf{u}^0(t, \mathbf{x})$ is a stabilizing control with a guaranteed quality estimate of $P(t_0, \mathbf{x}_0) = V(t_0, \mathbf{x}_0)$ and the unperturbed motion $\mathbf{x} = 0$ is uniformly asymptotically stable.

The proof of this theorem is based on Lyapunov's theorem on asymptotic stability and resembles that of Krasovskii's theorem on optimal stabilization [2].

Using the method of limit functions and equations [4], one can relax the conditions on $V(t, \mathbf{x})$ and $W(t, \mathbf{x}, \mathbf{u}(t, \mathbf{x}))$. Namely, $W(t, \mathbf{x}, \mathbf{u}(t, \mathbf{x}))$ can be of constant sign, $W(t, \mathbf{x}, \mathbf{u}(t, \mathbf{x})) \geq 0$, and for $V(t, \mathbf{x})$ one can remove the condition that the function should admit of an infinitesimal upper bound.

Theorem 2. Suppose that a Lyapunov function $V(t, \mathbf{x}) \in C^1(G)$ and a control function $\mathbf{u} = \mathbf{u}^0(t, \mathbf{x}) \in U$ exist for system (1.1) with control quality estimate (1.2) such that

1. $V(t, \mathbf{x})$ is positive definite and admits of an infinitesimal upper bound $\alpha_1(\|\mathbf{x}\|) \leq V(t, \mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|)$;
2. $B[V, t, \mathbf{x}, \mathbf{u}^0(t, \mathbf{x})] \leq 0$;
3. the right-hand side $(\mathbf{X}^0(t, \mathbf{x}) = \mathbf{X}(t, \mathbf{x}, \mathbf{u}^0(t, \mathbf{x}))$ of (1.1) and $W^0(t, \mathbf{x}) = W(t, \mathbf{x}, \mathbf{u}^0(t, \mathbf{x}))$ satisfy (2.1) and (2.2);
4. for any limit pair (Φ, Ω) corresponding to $(\mathbf{X}^0, W^0)$ the set $\{\Omega(t, \mathbf{x}) = 0\}$ does not contain any solutions of the limit system $\dot{\mathbf{x}} = \Phi(t, \mathbf{x})$, other than $\mathbf{x} = 0$.

Then $\mathbf{u}^0(t, \mathbf{x})$ is a stabilizing control function with a guaranteed estimate of quality $P(t, \mathbf{x}) = V(t, \mathbf{x})$. The unperturbed motion $\mathbf{x} = 0$ is uniformly asymptotically stable.

Proof. By (1.1) and condition 2 of the theorem, for any function $V(t, \mathbf{x})$ we have

$$dV/dt \leq -W(t, \mathbf{x}, \mathbf{u}^0(t, \mathbf{x})) \leq 0 \tag{3.1}$$

Taking into account that the function $W(t, \mathbf{x}, \mathbf{u}) \geq 0$ and that conditions 1, 3 and 4 are satisfied, by the theorem on asymptotic stability [4] we find that the solution $\mathbf{x} = 0$ of (1.1) is asymptotically stable.

By the asymptotic stability of $\mathbf{x} = 0$ and condition 1 of the theorem we have $\lim V(T, \mathbf{x}(T)) = 0$ as $T \to +\infty$. Integrating (3.1) from t_0 to T and taking the limit as $T \to \infty$, we obtain

$$\int_{t_0}^{\infty} W(t, \mathbf{x}^0_{(t)}, \mathbf{u}^0(t))dt \leq V(t_0, \mathbf{x}_0) = P(t_0, \mathbf{x}_0) \tag{3.2}$$

The theorem is proved.

If the identity

$$B[V, t, \mathbf{x}, \mathbf{u}^0(t, \mathbf{x})] \equiv 0 \tag{3.3}$$

is required in Theorem 2 in place of the second condition, and the condition

$$B[V, t, \mathbf{x}, \mathbf{u}^0(t, \mathbf{x})] \leq B[V, t, \mathbf{x}, \mathbf{u}^*(t, \mathbf{x})] \tag{3.4}$$

is added for any other control function $\mathbf{u}^*(t, \mathbf{x}) \in U$, then we have a theorem optimal stabilization [5].

Theorem 3. For (1.1) with (1.2), that is, min I for $\mathbf{u} \in U$, let a Lyapunov function $V(t, \mathbf{x}) \in C^1(G)$ and a control function $\mathbf{u} = \mathbf{u}^0(t, \mathbf{x}) \in U$ exist such that conditions 1, 3 and 4 of Theorem 2 and conditions (3.3) and (3.4) are satisfied.

Then $\mathbf{u}^0(t, \mathbf{x})$ is a stabilizing control function that solves the optimal stabilization problem for (1.1). Also, the unperturbed motion $\mathbf{x} = 0$ of (1.1) is uniformly asymptotically stable and

$$I^0 = \int_{t_0}^{\infty} W(t, \mathbf{x}^0_{(t)}, \mathbf{u}^0(t))dt = \min \int_{t_0}^{\infty} W(t, \mathbf{x}, \mathbf{u}^*(t, \mathbf{x}))dt = V(t_0, \mathbf{x}_0)$$

for any $\mathbf{u}^*(t, \mathbf{x}) \in U$ that solves the stabilization problem for the unperturbed motion $\mathbf{x} = 0$ of (1.1).

Theorem 4. Let $V(t, \mathbf{x}) \in C_1(G)$, $V(t, 0) = 0$ be a Lyapunov function and $\mathbf{u} = \mathbf{u}^0(t, \mathbf{x}) \in U$ a control function for (1.1) with quality criterion (2.1) such that conditions 2 and 3 of Theorem 2 and the following conditions are satisfied:

1. $V(t, \mathbf{x})$ is positive definite and $V(t, \mathbf{x}) \geq \alpha_1(\| \mathbf{x} \|)$;
2. there are numbers H_0 and H_1 $(0 < H_0 < H_1)$ such that $\sup(V(t, \mathbf{x})$ when $\| \mathbf{x} \| < H_0) < \alpha_1(H_1)$;
3. there is at least one sequence $t_n \to +\infty$ for which the limit pair (Φ, Ω) corresponding to $(\mathbf{X}^0, W^0)$ and the corresponding set $V_\infty^{-1}(t, c)$ are such that for any $c = c_0 = \text{const} > 0$ the set $\{V_\infty^{-1}(t, c): c = c_0\} \cap \{\Omega(t, \mathbf{x}) = 0\}$ contains no solutions of the limit system $\dot{\mathbf{x}} = \Phi(t, \mathbf{x})$.

Then $\mathbf{u}^0(t, \mathbf{x})$ is a stabilizing control with guaranteed quality estimate $P(t, \mathbf{x}) = V(t, \mathbf{x})$ and the unperturbed motion $\mathbf{x} = 0$ is asymptotically stable uniformly in $\mathbf{x}_0$.

Proof. By (1.1) and condition 2 of Theorem 2 we have inequality (3.1) for the derivative of $V(t, \mathbf{x})$. By conditions 1 and 2 it follows that the solution $\mathbf{x} = 0$ of (1.1) is stable. The solutions of (1.1) from the domain $\Gamma_1 = \{\| \mathbf{x} \| < H_0\}$ will be bounded, $\| \mathbf{x}(t, t_0, \mathbf{x}_0) \| \leq H_1$ for all $t \geq t_0$.

Let $\mathbf{x} = \mathbf{x}(t, t_0, \mathbf{x}_0)$ be a solution of (1.1) in Γ_1. By condition 1 of the theorem and (3.1) $V(t) = V(t, \mathbf{x}(t, t_0, \mathbf{x}_0)) \to c_0$ as $t \to +\infty$. Let $t_n \to +\infty$ be a sequence determining (Φ, Ω) and $V_\infty^{-1}(t, c)$ such that $\mathbf{x}(t_n) \to \mathbf{x}^*$ as $t_n \to +\infty$. We form the sequence of functions $\mathbf{x}_n(t) = \mathbf{x}(t_n + t, t_0, \mathbf{x}_0)$. By [1] the sequence of functions $\{\mathbf{x}_n(t) = \mathbf{x}(t_n + t, t_0, \mathbf{x}_0)\}$, which is defined for $t_n \geq t_0$, will converge to a solution $\mathbf{x} = \varphi(t) :]-\infty, +\infty[\to \Gamma$ of the system $\dot{\mathbf{x}} = \Phi(t, \mathbf{x})$ uniformly in each interval $[-T, T]$. Taking the limit as $t_k \to +\infty$, as in [4], we obtain

$$\varphi(t) \in \{\Omega(t, \mathbf{x}) = 0\} \cap \{V_\infty^{-1}(t, c) : c = c_0\}$$

But by condition 3 of the theorem this is possible only if $c_0 = 0$. So, along each solution $\mathbf{x}(t, t_0, \mathbf{x}_0)$: $\mathbf{x}_0 \in \Gamma_1$ of (1.1)

$$V(t, \mathbf{x}(t, t_0, \mathbf{x}_0)) \to 0 \quad \text{as} \quad t \to +\infty \tag{3.5}$$

It follows that the solution $\mathbf{x} = 0$ is asymptotically stable uniformly with respect to $\mathbf{x}_0$ [6].

Integrating (3.1) from t_0 to $+\infty$ and taking (3.5) into account, we obtain (3.2). The theorem is proved.

Example 1. We consider a linear controlled system, which is often encountered in robotics and is of practical interest, described by the equations

$$\ddot{\mathbf{x}} + P(t)\mathbf{x} = S(t)\mathbf{u}, \quad \mathbf{x} \in R^n, \quad \mathbf{u} \in R^n \tag{3.6}$$

where $P(t)$ is a bounded symmetric positive definite $(n \times n)$-matrix, $0 < P_0 \leqslant P(t) \leqslant P_1$, and $S(t)$ is a bounded $(n \times n)$-matrix of control actions $\mathbf{u}$.

Suppose that the quality of the transient is estimate by the functional (1.2), for which

$$W(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{u}) = \dot{\mathbf{x}}^T F(t)\dot{\mathbf{x}} + \mathbf{u}^T Q(t)\mathbf{u} \tag{3.7}$$

where $Q(t)$ is a bounded symmetric positive definite $(n \times n)$-matrix and the bounded non-negative matrix $F(t)$ satisfies the inequality

$$F \leqslant \frac{1}{4} P^{-1} S Q^{-1} S^T P^{-1} + \frac{1}{2} P^{-1} \dot{P} P^{-1} \tag{3.8}$$

Using the Lyapunov function

$$V(t, \mathbf{x}, \dot{\mathbf{x}}) = \frac{1}{2} \dot{\mathbf{x}}^T P^{-1}(t) \dot{\mathbf{x}} + \frac{1}{2} \mathbf{x}^T E \mathbf{x}$$

(where E is the identity matrix and $P^{-1}(t)$ is the inverse matrix to $P(t)$), using Theorem 2 we find that the control actions

$$\mathbf{u}^0(t, \mathbf{x}, \dot{\mathbf{x}}) = -\frac{1}{2} Q^{-1}(t) S^T(t) P^{-1}(t) \dot{\mathbf{x}}$$

subject to the condition

$$P^{-1} S Q^{-1} S^T P^{-1} + P^{-1} \dot{P} P^{-1} \geqslant \gamma_0 E \quad (\gamma_0 = \text{const} > 0)$$

solve for (3.6) the problem of stabilizing the null state $\dot{\mathbf{x}} = \mathbf{x} = 0$ with guaranteed estimate $V(t_0, \mathbf{x}_0)$ of functional (1.2) defined by (3.7) and (3.8).

But if F is defined by $4F = P^{-1} S Q^{-1} S^T P^{-1} + 2P^{-1} \dot{P} P^{-1}$, these action solve the corresponding problem of optimal stabilization.

Example 2. We consider the problem of using the gravitational moment to stabilize the controlled plane rotational motion of a satellite in an elliptic orbit. The equation of motion can be written in the form [7]

$$\alpha'' - l(\nu)\alpha' + m(\nu)\sin\alpha = 2l(\nu) + U \quad (\alpha = 2\theta)$$
$$l(\nu) = \frac{2e\sin\nu}{1 + e\cos\nu}, \quad m(\nu) = \frac{n^2}{1 + e\cos\nu}, \quad n^2 = 3\frac{A - C}{B} \tag{3.9}$$

where θ is the angle between Oz and Oz_0 and ν is the true anomaly of the centre of mass O of the satellite in an elliptic orbit with eccentricity e, $0 < e < 1$. The derivative with respect to ν is denoted by a prime; U is the control moment, $Ox_0y_0z_0$ is the orbital system of coordinates, and Ox, Oy and Oz are the principal central axes of inertia of the satellite with moments A, B and C. During the motion the axis Oy of the satellite remains constant and coincides with the normal axis Oy_0 to the orbit plane.

Let U be a control action that ensures the given rotational motion $\alpha = \alpha_0(\nu)$, $|\alpha_0(\nu)| \leqslant \pi - \delta$, $\delta > 0$. We define an additional control action $u = U - U_0$ so that the motion is asymptotically stable and the guaranteed estimate of quality

$$I = \int_{\nu_0}^{+\infty} (k_1 x'^2 + u^2) d\nu, \quad k_1(\nu) \geqslant 0, \quad x = \alpha - \alpha_0(\nu) \tag{3.10}$$

holds for the transient. Putting $u = -k(\nu)x'$, we obtain from (3.9) the equation of motion in terms of perturbations

$$x'' + (k(\nu) - l(\nu))x' + g(\nu, x)\sin\frac{x}{2} = 0, \quad g(\nu, x) = 2m(\nu)\cos\frac{\alpha_0(\nu) + x}{2} \tag{3.11}$$

We seek a solution using the Lyapunov function

$$V(\nu, x) = \frac{1}{2}\frac{1}{g(\nu, x)}x'^2 + 2(1 - \cos\tfrac{x}{2})$$

which is positive definite, since $g(\nu, x) \geqslant g_0 > 0$, and admits of an infinitesimal upper bound. By Theorem 2 we find that the equation $u^0 = -k(\nu)x'$ ensures the stabilization of the plane rotational motion $\alpha = \alpha_0(\nu)$ of the satellite about the centre of mass in an elliptic orbit with a guaranteed estimate of control quality $V(\nu_0, x_0)$ by means of (3.10) if

$$(4k(\nu) - 3l(\nu))\cos\frac{\alpha_0(\nu)}{2} \geqslant \gamma_0 + \alpha_0'(\nu)\sin\frac{\alpha_0(\nu)}{2}$$

$$k_1 + (k(\nu) - r(\nu))^2 \leq r(\nu)\left(r(\nu) - \frac{1}{4}\operatorname{tg}\frac{\alpha_0(\nu)}{2}\alpha_0'(\nu) - \frac{3}{2}l(\nu)\right) - \varepsilon_0, \quad r(\nu) = \frac{1}{2g(\nu, 0)}$$

where γ_0 and ε_0 are positive constants.

4. DETERMINATION OF THE FORM OF THE INTEGRAND IN THE QUALITY CRITERION AND THE CONTROL ACTIONS IN THE PROBLEM WITH ADDITIONAL CONTROL FORCES

We consider the system of equations

$$\dot{\mathbf{x}} = \mathbf{X}(t, \mathbf{x}), \quad \mathbf{x} \in R^n; \quad \mathbf{X}(t, 0) = 0 \tag{4.1}$$

where $\mathbf{X}(t, \mathbf{x})$ is a continuous function that satisfies the conditions for the existence and uniqueness of solutions in G.

Let there be a Lyapunov function $V(t, \mathbf{x}) \in C^1(G)$, $V(t, 0) = 0$ determining the stability of the solution $\mathbf{x} = 0$ of (4.1) and having a derivative $\dot{V}(t, \mathbf{x}) \leqslant 0$ by (4.1).

Suppose that additional control forces of the form $X_1(t, \mathbf{x}, \mathbf{u}) = M(t, \mathbf{x})\mathbf{u}$ are applied to system (4.1), where $\mathbf{u} \in R^r$, $M(t, \mathbf{x})$ is an $(n \times r)$-matrix, and the estimate of the integral (1.2) is a control quality estimate for the resulting control system

$$\dot{\mathbf{x}} = \mathbf{X}(t, \mathbf{x}) + M(t, \mathbf{x})\mathbf{u} \tag{4.2}$$

We pose the following problem: it is required to construct the form of the integrand $W(t, \mathbf{x}, \mathbf{u})$ for which the Lyapunov function $V(t, \mathbf{x})$ given for system (4.1) can define stabilization for (4.2) with a guaranteed quality estimate of $P(t, \mathbf{x}) \equiv V(t, \mathbf{x})$ of the control system (4.2).

This problem is similar to the optimal stabilization problem formulated and solved by Rumyantsev [8] as a development of problems concerned with the analytic construction of results [9] and the selection of an optimizing functional [10].

Following [8], we represent the integrand in (1.2) in the form

$$W(t, \mathbf{x}, \mathbf{u}) = F(t, \mathbf{x}) + \mathbf{u}^T R(t, \mathbf{x})\mathbf{u} \tag{4.3}$$

where $R(t, \mathbf{x})$ is a symmetric positive definite $(r \times r)$-matrix and $F(t, \mathbf{x})$ is a non-negative function to be determined.

Substituting $V(t, \mathbf{x})$ and $W(t, \mathbf{x}, \mathbf{u})$ into $B[V, t, \mathbf{x}, \mathbf{u}(t, \mathbf{x})]$, we can determine the control action

$$\mathbf{u}^0(t, \mathbf{x}) = -\frac{1}{2}R^{-1}(t, \mathbf{x})M^T(t, \mathbf{x})\left(\frac{\partial V}{\partial \mathbf{x}}(t, \mathbf{x})\right) \tag{4.4}$$

which makes $B[V, t, \mathbf{x}, \mathbf{u}]$ a minimum with respect to $\mathbf{u}$ (here $R^{-1}(t, \mathbf{x})$ is the inverse matrix to $R(t, \mathbf{x})$).

From the condition $B[V, t, \mathbf{x}, \mathbf{u}^0(t, \mathbf{x})] \leqslant 0$ we find relations which must be satisfied by $F(t, \mathbf{x})$

$$0 \leqslant F \leqslant -\dot{V} + \frac{1}{4}\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T MR^{-1}M^T\left(\frac{\partial V}{\partial \mathbf{x}}\right) \tag{4.5}$$

These relations provide a wider choice of the functional (1.2) with $W(t, \mathbf{x}, \mathbf{u})$ of the form (4.3) as compared to the optimal stabilization problem [7] when $F(t, \mathbf{x})$ is defined by the strict equality

$$F = -\dot{V} + \frac{1}{4}\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T MR^{-1}M^T\left(\frac{\partial V}{\partial \mathbf{x}}\right) \tag{4.6}$$

By (4.2) with $\mathbf{u}^0(t, \mathbf{x})$ from (4.4) the derivative of $V(t, \mathbf{x})$ will be computed from the formula

$$\frac{dV}{dt} = -W^0 = \dot{V} - \frac{1}{2}\left(\frac{\partial V}{\partial \mathbf{x}}\right)^T MR^{-1}M^T\left(\frac{\partial V}{\partial \mathbf{x}}\right)$$

Applying Theorem 2, we obtain the following result.

Theorem 5. Suppose that a positive definite function $V(t, \mathbf{x})$ having derivative $\dot{V}(t, \mathbf{x}) \leqslant 0$ and admitting of an infinitesimal upper bound is known for system (4.1) and the following conditions are satisfied:

1. the right-hand side $\mathbf{X}^0(t, \mathbf{x}) = \mathbf{X}(t, \mathbf{x}) + M(t, \mathbf{x})\mathbf{u}^0(t, \mathbf{x})$ in (4.2) and $W^0(t, \mathbf{x})$ satisfy (2.1) and (2.2);
2. for any limit pair (Φ, Ω) corresponding to $(\mathbf{X}(t, \mathbf{x}), W^0(t, \mathbf{x}))$ the set $\{\Omega(t, \mathbf{x}) = 0\}$ contains no solutions of the limit system $\dot{\mathbf{x}} = \Phi(t, \mathbf{x})$ except $\mathbf{x} = 0$.

Then the control action (4.4) is stabilizing with a guaranteed estimate of quality $V(t_0, \mathbf{x}_0)$ of the functional (1.2) with (4.3) and (4.5).

Corollary. If it is required in Theorem 5 that $B[V, t, \mathbf{x}, \mathbf{u}^0(t, \mathbf{x})] = 0$ and $B[V, t, \mathbf{x}, \mathbf{u}(t, \mathbf{x})] \geqslant 0$ for any other control action $\mathbf{u}(t, \mathbf{x}) \in U$, the remaining conditions being retained, then we have a solution of the problem of optimal stabilization of system (4.2) by a control action (4.4) with the minimum of the functional (1.2), (4.3), (4.6).

Example. We consider a holonomic mechanical system described by Lagrange's equations of the second kind

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = 0 \tag{4.7}$$

for which the structure of the Lagrange function is $L(t, \mathbf{q}, \dot{\mathbf{q}}) = L_2 + L_1 + L_0$, where $L_2 = \dot{\mathbf{q}}^T A(q)\dot{\mathbf{q}}/2$ is a quadratic form of the velocities $\dot{\mathbf{q}}$, $L_1 = BT(\mathbf{q})\dot{\mathbf{q}}$ is a linear form of the velocities, $\dot{\mathbf{q}}$, and $L_0 = L_0(t, \mathbf{q})$ $(L_0(t, 0) \equiv 0)$ satisfies the following conditions

$$\partial L_0/\partial \mathbf{q} = 0 \quad \text{for} \quad \mathbf{q} = 0, \quad \partial L_0/\partial t \geqslant 0, \quad -L_0(t, \mathbf{q}) \geqslant \alpha(\|\mathbf{q}\|)$$

Such a system has a position of equilibrium $\mathbf{q} = \dot{\mathbf{q}} = 0$, which is stable because $(L_2 - L_0)^{\cdot} = -\partial L_0/\partial t \leqslant 0$.

We now formulate the problem of determining forces of the form $Q = M(t, \mathbf{q}, \dot{\mathbf{q}})\mathbf{u}$ ($M(t, \mathbf{q}, \dot{\mathbf{q}})$ is a bounded ($n \times r$)-matrix) such that for the controlled system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = M(t, \mathbf{q}, \dot{\mathbf{q}})\mathbf{u} \tag{4.8}$$

1. the equilibrium $\mathbf{q} = \dot{\mathbf{q}} = 0$ is asymptotically stable;
2. there is a guaranteed estimate of the quality of the control (1.2) with integrand

$$W(t, \mathbf{q}, \dot{\mathbf{q}}) = F(t, \mathbf{q}, \dot{\mathbf{q}}) + \mathbf{u}^T R(t, \mathbf{q}, \dot{\mathbf{q}})\mathbf{u} \tag{4.9}$$

where $R(t, \mathbf{q}, \dot{\mathbf{q}})$ is a bounded symmetric positive definite $(r \times r)$-matrix and $F(t, \mathbf{q}, \dot{\mathbf{q}})$ is a non-negative function to be determined.

Using $V = L_2 - L_0$ as Lyapunov's function, by the results of Section 4 we have the control action

$$\mathbf{u}^0(t, \mathbf{q}, \dot{\mathbf{q}}) = -\frac{1}{2}R^{-1}(t, \mathbf{q}, \dot{\mathbf{q}})M^T(t, \mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \tag{4.10}$$

which solves the problem, along with the following conditions for choosing $F(t, \mathbf{q}, \dot{\mathbf{q}})$

$$0 \leqslant F(t, \mathbf{q}, \dot{\mathbf{q}}) \leqslant \frac{1}{4}\dot{\mathbf{q}}^T N(t, \mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}, \quad N = MR^{-1}M^T \tag{4.11}$$

Furthermore, taking (4.10) into account, we have the bound

$$\frac{dV}{dt} \leqslant -\frac{1}{2}\dot{\mathbf{q}}^T N(t, \mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} \leqslant 0$$

for the time derivative of V. For $\mathbf{u} = \mathbf{u}^0(t, \mathbf{q}, \dot{\mathbf{q}})$ the limit system corresponding to (4.8) has the form [4]

$$\frac{d}{dt}\left(\frac{\partial L_2}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L_2}{\partial \mathbf{q}} = \frac{\partial L_{0^*}}{\partial \mathbf{q}} + N^*\dot{\mathbf{q}} + G\dot{\mathbf{q}}, \quad G^T = -G \tag{4.12}$$

where L_0^* and N^* are the limit function and matrix corresponding to L_0 and N.

Suppose that $\{\dot{\mathbf{q}}^T N^*(t, \mathbf{q}, \dot{\mathbf{q}})\,\dot{\mathbf{q}} = \mathbf{0}\}$ contains no solutions of (4.12) except $\mathbf{q} = \dot{\mathbf{q}} = 0$. Then by Theorem 5 the control action (4.10) solves the stabilization problem for (4.8) with a guaranteed estimate of the control quality $P(t, \mathbf{q}, \dot{\mathbf{q}}) = L_2(\mathbf{q}, \dot{\mathbf{q}}) - L(t, \mathbf{q})$ by the functional (1.2), the integrand of which can be chosen from (4.9) and (4.11).

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